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On the Precision of Uniform Approximation of Continuous Functions by Certain Linear Positive Operators of Convolution Type

R. BOJANIC*

Department of Mathematics, The Ohio State University, Columbus, Ohio 43210

AND

O. SHISHA

Aerospace Research Laboratories, Wright-Patterson AFB, Ohio 45433†

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1.

Let φ be a nonnegative, even and continuous function on $[-r, r]$, decreasing on $[0, r]$ and such that $\varphi(0) = 1$ and $0 \leqslant \varphi(t) < 1$ for $0 < t \leqslant r$.

For a continuous function f on $I = [a, b]$ with $b - a \leqslant r$, let

$$K_n(f, x) = \rho_n \int_a^b f(t)\, \varphi^n(t - x)\, dt, \qquad n = 1, 2,..., \tag{1.1}$$

where

$$1/\rho_n = 2 \int_0^r \varphi^n(t)\, dt.$$

Linear positive operators of this form were introduced by Korovkin in his book "Linear Operators and Approximation Theory." He has proved that

$$\lim_{n \to \infty} K_n(f, x) = f(x),$$

uniformly on every interval $I_\delta = [a + \delta, b - \delta]$, where $0 < \delta < \frac{1}{2}(b - a)$.

Many special, well known linear positive operators are of essentially this form.

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† Present address: Mathematics Research Center, Code 7840, Naval Research Laboratory, Washington, D.C. 20390.

We have, for instance,

$$\varphi(t) = e^{-t^2}, \qquad 0 < r < \infty \qquad \text{(Weierstrass [1]);}$$

$$\varphi(t) = 1 - t^2, \qquad r = 1 \qquad \text{[Landau [2]);}$$

$$\varphi(t) = 1 - t^{2k}, \qquad r = 1,\ k = 1, 2, \ldots \qquad \text{(Mamedov [3]);}$$

$$\varphi(t) = e^{-|t|}, \qquad 0 < r < \infty \qquad \text{(Picard);}$$

$$\varphi(t) = e^{-|t|^{1/k}}, \qquad 0 < r < \infty \qquad \text{(Bui, Fedorov, Červakov [4]);}$$

$$\varphi(t) = \cos^2(t/2), \qquad r = \pi \qquad \text{(de la Vallée-Poussin [5]);}$$

$$\varphi(t) = 1/I_0(x), \qquad 0 < r < \infty \qquad \text{(Mirakian [6]).}$$

Here $I_0(x) = \sum_{k=0}^{\infty} (x/2)^{2k}/(2k)$ is the Bessel function of imaginary argument. Mirakian has also studied linear positive operators generated by $\varphi(t) = 1/\psi(t)$, where $\psi(t) = 1 + \sum_{k=1}^{\infty} c_k t^{2k}$, assuming that all the coefficients c_k are positive and that the series converges on $[-r, r]$.

The aim of this paper is to study the degree of approximation of f by linear positive operators $K_n(f)$. Using an inequality of Shisha and Mond (see [7, 8]) we shall prove first the following result.

For $n = 1, 2, \ldots$, *we have*

$$\| K_n(f) - f \|_{I_\delta} \leq 2\omega_f(\mu_n) + \| f \|_I \, \delta^{-2}\mu_n^2 \tag{1.2}$$

where

$$\mu_n^2 = \frac{\int_0^r t^2 \varphi^n(t)\, dt}{\int_0^r \varphi^n(t)\, dt} .$$

Here $\| g \|_E = \sup\{ |g(x)| : x \in E\}$, and ω_f is the modulus of continuity of f.

The degree of approximation thus depends on how fast the sequence (μ_n) converges to zero. We shall show here that this depends on the asymptotic behavior of the function φ in the neighborhood of zero. Generally speaking, the faster $\varphi(x)$ approaches 1 as $x \to 0$, the slower μ_n approaches 0 as $n \to \infty$. More precisely, we have the following result:

If

$$\lim_{x \to 0+} \frac{1 - \varphi(x)}{x^\alpha} = c$$

where $0 < c < \infty$ *and* $0 < \alpha < \infty$, *then*

$$\mu_n = \mathcal{O}(n^{-1/\alpha}), \qquad (n \to \infty). \tag{1.3}$$

From (1.2) and (1.3) we obtain immediately our first main result:

THEOREM 1. *Let φ be a nonnegative, even and continuous function on $[-r, r]$, decreasing on $[0, r]$, such that $\varphi(0) = 1$ and $0 \leqslant \varphi(x) < 1$ if $0 < x \leqslant r$. For every $f \in C[a, b]$, $0 < b - a \leqslant r$, let $K_n(f)$ be defined by (1.1). If, for some $\alpha > 0$ and $c > 0$,*

$$\lim_{x \to 0+} \frac{1 - \varphi(x)}{x^{\alpha}} = c,$$

then there exist positive numbers $L(\varphi)$, $M(\varphi)$ and $N(\varphi)$ such that

$$\| K_n(f) - f \|_{I_\delta} \leqslant L(\varphi)\, \omega_f(n^{-1/\alpha}) + M(\varphi) \| f \|_I\, \delta^{-2} n^{-2/\alpha} \tag{1.4}$$

for every $n \geqslant N(\varphi)$.

As corollaries of Theorem 1 we obtain the following results valid for $x \in [a + \delta, b - \delta]$ and $n \geqslant N(\varphi)$:

If φ is the kernel of Weierstrass, Landau, de la Vallée-Poussin or Mirakian, we have

$$\lim_{x \to 0} \frac{1 - \varphi(x)}{x^2} = c, \qquad 0 < c < \infty,$$

and so

$$\| K_n(f) - f \|_{I_\delta} \leqslant L(\varphi)\, \omega_f(n^{-1/2}) + M(\varphi) \| f \|_I\, \delta^{-2} n^{-1}.$$

If φ is the kernel of Mamedov, we have

$$\lim_{x \to 0} \frac{1 - \varphi(x)}{x^{2k}} = 1,$$

and so

$$\| K_n(f) - f \|_{I_\delta} \leqslant L(\varphi)\, \omega_f(n^{-1/2k}) + M(\varphi) \| f \|_I\, \delta^{-2} n^{-1/k}.$$

If φ is the kernel of Picard or, more generally, of Bui, Fedorov and Červakov, then

$$\lim_{x \to 0+} \frac{1 - \varphi(x)}{x^{1/k}} = 1$$

and, consequently,

$$\| K_n(f) - f \|_{I_\delta} \leqslant L(\varphi)\, \omega_f(n^{-k}) + M(\varphi) \| f \|_I\, \delta^{-2} n^{-2k}.$$

Finally, we shall show that Theorem 1 cannot be essentially improved in the class $C_\Omega[a, b]$ of continuous functions f on $[a, b]$ which have the property that $\omega_f(h) \leqslant \Omega(h)$ for every $h \geqslant 0$. Here $\Omega(\neq 0)$ is a fixed modulus of continuity, i.e., a continuous, increasing and subadditive function on $[0, \infty)$ with $\Omega(0) = 0$.

Supposing that φ satisfies the same hypotheses as in Theorem 1, we have as our second main result the following.

THEOREM 2. *Let*

$$\Delta_n(\Omega) = \sup\{\|K_n(f) - f\|_{I_\delta} : f \in C_\Omega[a, b] \text{ and } \|f\|_I \leqslant 1\}.$$

Then there exist positive numbers $p(\varphi, \Omega)$, $P(\varphi, \Omega)$ and $N(\varphi, \Omega)$ such that

$$0 < p(\varphi, \Omega) \leqslant \frac{\Delta_n(\Omega)}{\Omega(n^{-1/\alpha})} \leqslant \delta^{-2}P(\varphi, \Omega) < \infty \tag{1.5}$$

for all $n \geqslant N(\varphi, \Omega)$.

From Theorem 2 we can obtain immediately the following corollaries.

COROLLARY 1. *For every function $f \in C_\Omega[a, b]$ with $\|f\|_I \leqslant 1$, we have*

$$\|K_n(f) - f\|_{I_\delta} \leqslant \delta^{-2}P(\varphi, \Omega)\,\Omega(n^{-1/\alpha}) \tag{1.6}$$

for all $n \geqslant N(\varphi, \Omega)$, and the sequence $(\Omega(n^{-1/\alpha}))$ cannot be replaced by any sequence (Γ_n) of positive numbers for which

$$\liminf_{n\to\infty} \frac{\Gamma_n}{\Omega(n^{-1/\alpha})} = 0.$$

To see this, suppose that there were a positive number Q such that we had

$$\|K_n(f) - f\|_{I_\delta} \leqslant Q\Gamma_n$$

for every $f \in C_\Omega[a, b]$ with $\|f\|_I \leqslant 1$, and all $n \geqslant N$. Then, by Theorem 2, we would have, for all $n \geqslant \max(N, N(\varphi, \Omega))$, the inequality

$$Q\Gamma_n \geqslant \Delta_n(\Omega) \geqslant p(\varphi, \Omega)\,\Omega(n^{-1/\alpha}),$$

and so

$$\liminf_{n\to\infty} \frac{\Gamma_n}{\Omega(n^{-1/\alpha})} \geqslant \frac{p(\varphi, \Omega)}{Q} > 0.$$

For some functions Ω, such as $\Omega(h) = h^\sigma$, $0 < \sigma \leqslant 1$, we can make a slightly stronger statement.

COROLLARY 2. *Let Ω be a decreasing, continuous and subadditive function on $[0, \infty)$, with $\Omega(0) = 0$, such that*

$$\lim_{h \to 0+} \frac{\Omega(Ah)}{\Omega(h)} = A^{\sigma}, \qquad 0 < \sigma \leqslant 1, \tag{1.7}$$

for every $A > 0$. Then (1.6) holds for every $f \in C_\Omega[a, b]$ with $\|f\|_I \leqslant 1$, and the sequence $(n^{-1/\alpha})$ cannot be replaced by any sequence (γ_n) of positive numbers such that

$$\liminf_{n \to \infty} n^{1/\alpha}\gamma_n = 0.$$

This result, too, is a very simple consequence of Theorem 2. Assuming that there exists a positive number Q such that

$$\| K_n(f) - f \|_{I_\delta} \leqslant Q\Omega(\gamma_n),$$

for every $f \in C_\Omega[a, b]$ with $\|f\|_I \leqslant 1$, we find, by Theorem 2, that

$$Q\Omega(\gamma_n) \geqslant \Delta_n(\Omega) \geqslant p(\varphi, \Omega)\, \Omega(n^{-1/\alpha}) = p(\varphi, \Omega)\, \Omega\left(\frac{\gamma_n}{n^{1/\alpha}\gamma_n}\right).$$

Let (n_k) be such that $n_k^{1/\alpha}\gamma_{n_k} \to 0$ $(k \to \infty)$. Given an $M > (Q/p(\varphi, \Omega))^{1/\sigma}$, we can find an N_M such that

$$\frac{1}{n_k^{1/\alpha}\gamma_{n_k}} \geqslant M \qquad \text{for all } k \geqslant N_M .$$

We have then, by the monotonicity of Ω,

$$Q\Omega(\gamma_{n_k}) \geqslant p(\varphi, \Omega)\, \Omega(M\gamma_{n_k}) \qquad \text{for all } k \geqslant N_M ,$$

and so

$$Q \geqslant p(\varphi, \Omega) \lim_{k \to \infty} \frac{\Omega(M\gamma_{n_k})}{\Omega(\gamma_{n_k})} = p(\varphi, \Omega)\, M^{\sigma},$$

which is impossible, since $M^{\sigma} > Q/p(\varphi, \Omega)$.

Condition (1.7), although not the most general, is certainly necessary for the validity of Corollary 2. To show that Corollary 2 is false without Condition (1.7), consider the function

$$\Omega_0(h) = \begin{cases} 0, & h = 0, \\ \dfrac{1}{\log(1/h)}, & 0 < h \leqslant e^{-1}. \end{cases}$$

It is easy to see that Ω_0 is a modulus of continuity which does not satisfy Condition (1.7) since

$$\lim_{h\to 0+} \frac{\Omega_0(Ah)}{\Omega_0(h)} = 1$$

for every $A > 0$. This modulus of continuity does not distinguish asymptotically between the sequences (n^{-p}) and (n^{-q}) as far as the degree of convergence is concerned, since

$$\lim_{n\to\infty} \frac{\Omega_0(n^{-p})}{\Omega_0(n^{-q})} = \frac{q}{p}.$$

Consequently, in the estimate

$$\| K_n(f) - f \|_{I_\delta} \leq \delta^{-2} P(\varphi, \Omega_0)\, \Omega_0(n^{-1/\alpha})$$

we can replace the sequence $(n^{-1/\alpha})$ by any sequence (n^{-q}) with $q > 0$, without changing the degree of convergence. We have actually in this case the estimate

$$\| K_n(f) - f \|_{I_\delta} \leq \frac{\alpha\delta^{-2} P(\varphi, \Omega_0)}{\log n}$$

for all $f \in C_{\Omega_\delta}[a, b]$ with $\| f \|_I \leq 1$ and for all $n \geq N(\varphi, \Omega_0)$. However, in view of Corollary 1, the sequence $(1/\log n)$ cannot be replaced by any sequence (Γ_n) such that $\liminf_{n\to\infty} \Gamma_n \log n = 0$.

2.

The proofs of Theorems 1 and 2 are based on two lemmas.

LEMMA 1. *Let φ be a nonnegative, even and continuous function on $[-r, r]$. For every $f \in C[a, b]$, $0 < b - a < r$, let $K_n(f)$ be defined by* (1.1). *We have then, for $n = 1, 2,...,$*

$$\| K_n(f) - f \|_{I_\delta} \leq 2\omega_f(\mu_n) + \| f \|_I\, \delta^{-2} \mu_n^2,$$

where

$$\mu_n^2 = \frac{\int_0^r t^2 \varphi^n(t)\, dt}{\int_0^r \varphi^n(t)\, dt}.$$

Proof. Since K_n is a linear positive operator on $C[a,b]$, into $C[a,b]$, we can apply the inequality of Shisha and Mond [3] and obtain, for every $x \in [a,b]$,

$$|K_n(f, x) - f(x)| \leq (1 + K_n(1, x))\, \omega_f(\mu_n) + \| f \|_I\, | K_n(1, x) - 1 |, \quad (2.1)$$

where

$$\mu_n^2 \geqslant \max\{K_n((t-x)^2, x) : a \leqslant x \leqslant b\}.$$

We have, for every $x \in [a, b]$,

$$\begin{aligned} K_n(1, x) &= \rho_n \int_a^b \varphi^n(t-x)\,dt \\ &= \rho_n \int_{a-x}^{b-x} \varphi^n(t)\,dt \\ &\leqslant \rho_n \int_{-(b-a)}^{b-a} \varphi^n(t)\,dt. \end{aligned}$$

Since $b - a \leqslant r$, and φ is even, we have

$$K_n(1, x) \leqslant \rho_n \int_{-r}^{r} \varphi^n(t)\,dt = 1. \tag{2.2}$$

Next,

$$\begin{aligned} K_n(1, x) &= \rho_n \int_{a-x}^{b-x} \varphi^n(t)\,dt \\ &= \rho_n \int_{-r}^{r} \varphi^n(t)\,dt - \rho_n \int_{b-x}^{c} \varphi^n(t)\,dt - \rho_n \int_{-r}^{-(x-a)} \varphi^n(t)\,dt. \end{aligned}$$

Since the first term on the right side is 1, and since φ is even, it follows that for $x \in I_\delta = [a + \delta, b - \delta]$, $0 < \delta < \frac{1}{2}(b - a)$, we have

$$\begin{aligned} |K_n(1, x) - 1| &= \rho_n \int_{b-x}^{r} \varphi^n(t)\,dt + \rho_n \int_{x-a}^{r} \varphi^n(t)\,dt \\ &\leqslant 2\rho_n \int_{\delta}^{r} \varphi^n(t)\,dt \\ &\leqslant 2\delta^{-2}\rho_n \int_{\delta}^{r} t^2\varphi^n(t)\,dt. \end{aligned}$$

Hence, for $x \in I_\delta$, we have

$$|K_n(1, x) - 1| \leqslant 2\delta^{-2}\rho_n \int_0^r t^2\varphi^n(t)\,dt. \tag{2.3}$$

Finally, for $x \in I = [a, b]$, we have

$$\begin{aligned} K_n((t-x)^2, x) &= \rho_n \int_a^b (t-x)^2\,\varphi^n(t-x)\,dt \\ &= \rho_n \int_{a-x}^{b-x} t^2\varphi^n(t)\,dt \\ &\leqslant \rho_n \int_{-(b-a)}^{b-a} t^2\varphi^n(t)\,dt. \end{aligned}$$

Hence,

$$K_n((t-x)^2, x) \leqslant \rho_n \int_{-r}^{r} t^2\varphi^n(t)\,dt = \mu_n^{\,2}. \tag{2.4}$$

and Lemma 1 follows from (2.1)–(2.4).

LEMMA 2. *Let φ be a nonnegative and decreasing function on $[0, r]$, $\varphi(0) = 1$, $0 \leqslant \varphi(x) < 1$ if $0 < x \leqslant r$, and*

$$\lim_{x\to 0+} \frac{1-\varphi(x)}{x^\alpha} = c \tag{2.5}$$

where α and c are positive numbers. Then, for every $\beta \geqslant 0$ and $n = 1, 2,\ldots$ we have

$$B(\alpha,\beta)(nc)^{-(\beta+1)/\alpha} - (2c)^{-(\beta+1)/\alpha} e^{-2nc\eta^\alpha}$$
$$\leqslant \int_0^r t^\beta\varphi^n(t)\,dt \leqslant A(\alpha,\beta)(nc)^{-(\beta+1)/\alpha} + r^{\beta+1}e^{-nc\eta^\alpha/2}, \tag{2.6}$$

where $A(\alpha, \beta)$ and $B(\alpha, \beta)$ are positive numbers.

Proof. From (2.5) follows that we can find an $\eta_1 \in (0, (b-a)/2)$ such that

$$\frac{1}{2}c \leqslant \frac{1-\varphi(x)}{x^\alpha} \leqslant 2c,$$

whenever $0 < x \leqslant \eta_1 < r$. Let $\eta = \min\ (\eta_1, (1/2c)^{1/\alpha})$. Then $0 < \eta < (b-a)/2 < r$ and, for $0 < x \leqslant \eta$, we have

$$0 \leqslant 1 - 2cx^\alpha \leqslant \varphi(x) \leqslant 1 - \tfrac{1}{2}cx^\alpha.$$

Since φ is decreasing on $[0, r]$ and $\beta \geqslant 0$, we have

$$\int_0^r t^\beta\varphi^n(t)\,dt \leqslant \int_0^\eta t^\beta\varphi^n(t)\,dt + \varphi^n(\eta)\int_\eta^r t^\beta\,dt$$
$$\leqslant \int_0^\eta t^\beta\left(1-\frac{1}{2}ct^\alpha\right)^n dt + \frac{r^{\beta+1}}{\beta+1}\left(1-\frac{1}{2}c\eta^\alpha\right)^n.$$

Since $0 \leqslant 1 - \frac{1}{2}ct^\alpha \leqslant e^{-ct^\alpha/2}$, it follows that

$$\int_0^r t^\beta\varphi^n(t)\,dt \leqslant \int_0^\eta t^\beta e^{-nct^\alpha/2}\,dt + r^{\beta+1}e^{-nc\eta^\alpha/2}$$
$$\times \left(\frac{2}{nc}\right)^{(\beta+1)/\alpha}\int_0^{\eta(nc/2)^{1/\alpha}} x^\beta e^{-x^\alpha}\,dx + r^{\beta+1}e^{-nc\eta^\alpha/2},$$

and the right-hand side of (2.6) follows with

$$A(\alpha, \beta) = 2^{(\beta+1)/\alpha} \int_0^\infty x^\beta e^{-x^\alpha}\, dx.$$

Next,

$$\begin{aligned}\int_0^r t^\beta \varphi^n(t)\, dt &\geqslant \int_0^\eta t^\beta \varphi^n(t)\, dt \\ &\geqslant \int_0^\eta t^\beta (1 - 2ct^\alpha)^n\, dt \\ &\geqslant \frac{(2c)^{-(\beta+1)/\alpha}}{\alpha} \int_0^{2c\eta^\alpha} x^{((\beta+1)/\alpha)-1}(1 - x)^n\, dx.\end{aligned}$$

Since $2c\eta^\alpha \leqslant 1$, we have

$$\begin{aligned}&\int_0^r t^\beta \varphi^n(t)\, dt \\ &\quad\geqslant \frac{(2c)^{-(\beta+1)/\alpha}}{\alpha}\left(\int_0^1 x^{((\beta+1)/\alpha)-1}(1-x)^n\, dx - \int_{2c\eta^\alpha}^1 x^{((\beta+1)/\alpha)-1}(1-x)^n\, dx\right) \\ &\quad\geqslant \frac{(2c)^{-(\beta+1)/\alpha}}{\alpha} \cdot \frac{\Gamma((\beta+1)/\alpha)\, \Gamma(n+1)}{\Gamma(n+1+(\beta+1)/\alpha)} - \frac{(2c)^{-(\beta+1)/\alpha}}{\beta+1}(1 - 2c\eta^\alpha)^n.\end{aligned}$$

Now, for $1 < y \leqslant x$, we have

$$\frac{x^{x-1}e^y}{y^{y-1}e^x} \leqslant \frac{\Gamma(x)}{\Gamma(y)} \leqslant \frac{x^{x-1/2}e^y}{y^{y-1/2}e^x}$$

(see [9]). Using the last inequality, we find that

$$\begin{aligned}&\frac{\Gamma(n+1+(\beta+1)/\alpha)}{\Gamma(n+1)} \\ &\quad\leqslant e^{-(\beta+1)/\alpha}\left(n + \frac{\beta+1}{\alpha} + 1\right)^{(\beta+1)/\alpha}\left(1 + \frac{(\beta+1)/\alpha}{n+1}\right)^{n+1/2} \\ &\quad\leqslant n^{(\beta+1)/\alpha}\left(2 + \frac{\beta+1}{\alpha}\right)^{(\beta+1)/\alpha}.\end{aligned}$$

We have also $1 - 2c\eta^\alpha \leqslant e^{-2c\eta^\alpha}$. Hence,

$$\int_0^r t^\beta \varphi^n(t)\, dt \geqslant \frac{(2c)^{-(\beta+1)/\alpha}}{\alpha} n^{-(\beta+1)/\alpha}\left(2 + \frac{\beta+1}{\alpha}\right)^{-(\beta+1)/\alpha} - (2c)^{-(\beta+1)/\alpha} e^{-2nc\eta^\alpha},$$

and the left side of (2.6) follows with

$$B(\alpha, \beta) = \alpha^{-1} 2^{-(\beta+1)/\alpha} \left(2 + \frac{\beta + 1}{\alpha}\right)^{(\beta+1)/\alpha}.$$

Proof of Theorem 1. By Lemma 1 we have, for $n = 1, 2, \ldots$

$$\| K_n(f) - f \|_{I_\delta} \leqslant 2\omega_f(\mu_n) + \| f \|_I \, \delta^{-2} \mu_n^2,$$

where

$$\mu_n^2 = \frac{\int_0^r t^2 \varphi^n(t)\, dt}{\int_0^r \varphi^n(t)\, dt}.$$

By Lemma 2 we have, for all n sufficiently large,

$$\mu_n^2 \leqslant \frac{A(\alpha, 2)(nc)^{-3/\alpha} + r^3 e^{-nc\eta^\alpha/2}}{B(\alpha, 0)(nc)^{-1/\alpha} - (2c)^{-1/\alpha} e^{-2nc\eta^\alpha}}.$$

Hence,

$$\limsup_{n \to \infty} n^{1/2\alpha} \mu_n^2 \leqslant \frac{A(\alpha, 2)}{B(\alpha, 0)} c^{-2/\alpha} < \infty,$$

and Theorem 1 follows.

Proof of Theorem 2. For every $f \in C_\Omega[a, b]$ such that $\| f \|_I \leqslant 1$, we have by Theorem 1

$$\| K_n(f) - f \|_{I_\delta} \leqslant L(\varphi)\, \Omega(n^{-1/\alpha}) + M(\varphi)\, \delta^{-2} n^{-2/\alpha},$$

for every $n \geqslant N(\varphi)$. Hence, the inequality,

$$\Delta_n(\Omega) \leqslant L(\varphi)\, \Omega(h^{-1/\alpha}) + M(\varphi) \delta^{-2} n^{-1/\alpha},$$

holds for every $n \geqslant N(\varphi)$. Since Ω is a modulus of continuity $\not\equiv 0$, we can find a positive number c such that $\Omega(h) \geqslant ch$ for every $h \geqslant 0$. Hence, for every $n \geqslant N(\varphi)$ we have

$$\Delta_n(\Omega) \leqslant (L(\varphi) + M(\varphi)\, \delta^{-2} c^{-1})\, \Omega(n^{-1/\alpha})$$

and the right side of (1.5) follows.

Next, let

$$f(x) = \frac{\Omega(| x - (a + b)/2 |)}{\Omega((b - a)/2)}, \qquad x \in [a, b].$$

Since Ω is a modulus of continuity, it follows easily that $f \in C_\Omega[a, b]$ and that $\| f \|_I = 1$. Consequently, we have

$$\Delta_n(\Omega) \geq \| f \|_{I_\delta} \geq | K_n(f, (a + b)/2) - f((a + b)/2) |,$$

i.e.,

$$\Delta_n(\Omega) \geq K_n(f, (a + b)/2). \tag{2.7}$$

Using the definition of the operator K_n, we find that

$$\begin{aligned} K_n\left(f, \frac{a+b}{2}\right) &= \delta_n \int_a^b \Omega\left(\left| t - \frac{a+b}{2} \right|\right) \varphi^n\left(t - \frac{a+b}{2}\right) dt \\ &\geq \delta_n \int_{(a+b)/2}^b \Omega\left(t - \frac{a+b}{2}\right) \varphi^n\left(t - \frac{a+b}{2}\right) dt \\ &\geq \delta_n \int_0^{(b-a)/2} \Omega(t)\, \varphi^n(t)\, dt, \end{aligned}$$

where

$$\delta_n = \frac{\mu_n}{\Omega(b-a)/2} = \frac{1}{2\Omega((b-a)/2) \int_0^r \varphi^n(t)\, dt}. \tag{2.8}$$

Now, as in the proof of Lemma 2, we can find an $\eta \in (0, (b - a)/2)$ such that, for $0 < x \leq \eta$, we have

$$\varphi(x) \geq 1 - 2cx^\alpha \geq 0.$$

For $n \geq \eta^{-\alpha}$ we have $n^{-1/\alpha} \leq \eta < (b - a)/2$, and so

$$\begin{aligned} K_n\left(f, \frac{a+b}{2}\right) &\geq \delta_n \int_0^{n^{-1/\alpha}} \Omega(t)\, \varphi^n(t)\, dt \\ &\geq \delta_n \int_0^{n^{-1/\alpha}} \Omega(t)(1 - 2ct^\alpha)^n\, dt. \end{aligned}$$

Since $t \leq n^{-1/\alpha}$ and Ω is a modulus of continuity, we have

$$2\,\frac{\Omega(t)}{t} \geq n^{1/\alpha}\Omega(n^{-1/\alpha})$$

and so

$$K_n\left(f, \frac{a+b}{2}\right) \geq \frac{1}{2}\,\delta_n n^{1/\alpha}\Omega(n^{-1/\alpha}) \int_0^{n^{-1/\alpha}} t(1 - 2ct^\alpha)^n\, dt.$$

But

$$\int_0^{n^{-1/\alpha}} t(1-2ct^\alpha)^n\,dt \geq \left(1-\frac{2c}{n}\right)^n \int_0^{n^{-1/\alpha}} t\,dt = \frac{1}{2}n^{-2/\alpha}\left(1-\frac{2c}{n}\right)^n;$$

therefore, from the preceding inequality it follows that

$$K_n\left(f, \frac{a+b}{2}\right) \geq \frac{1}{4}n^{-1/\alpha}\,\delta_n\Omega(n^{-1/\alpha})\left(1-\frac{2c}{n}\right)^n \qquad \text{for} \quad n > \max(\eta^{-\alpha}, 2c).$$

From this inequality and (2.7) we get

$$\liminf_{n\to\infty} \frac{\Delta_n(\Omega)}{\Omega(n^{-1/\alpha})} \geq \frac{e^{-2c}}{4} \liminf_{n\to\infty} n^{-1/\alpha}\,\delta_n. \tag{2.9}$$

Finally, using (2.8) and Lemma 2, we find that

$$1/\delta_n = 2\Omega\left(\frac{b-a}{2}\right)\int_0^r \varphi^n(t)\,dt \leq 2\Omega\left(\frac{b-a}{2}\right)(A(\alpha,0)(nc)^{-1/\alpha} + re^{-ncr^\alpha/2}),$$

i.e.,

$$n^{-1/\alpha}\,\delta_n \geq \frac{1}{2\Omega((b-a)/2)(A(\alpha,0)\,c^{-1/\alpha} + rn^{1/\alpha}e^{-ncr^\alpha/2})},$$

and it follows that

$$\liminf_{n\to\infty} n^{-1/\alpha}\,\delta_n \geq \frac{c^{1/\alpha}}{2\Omega((b-a)/2)\,A(\alpha,0)}. \tag{2.10}$$

Combining (2.9) and (2.10), we see that

$$\liminf_{n\to\infty} \frac{\Delta_n(\Omega)}{\Omega(n^{-1/\alpha})} \geq \frac{c^{1/\alpha}e^{-2c}}{8\Omega((b-a)/2)\,A(\alpha,0)} > 0,$$

and Theorem 2 is proved.

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